

Dynamically Operating on Threshold Graphs and Related Classes (Extended Abstract)

Tiziana Calamoneri^{1,2} Angelo Monti³ Rossella Petreschi^{1,4}

*Computer Science Department,
"Sapienza" University of Rome, Italy*

1 Introduction

A graph $G = (V, E)$ is a *threshold graph* if there is a mapping $a : V \rightarrow \mathbb{R}^+$ and a positive real number S such that $a(v) < S$ for all $v \in V$ and $(v, w) \in E$ if and only if $a(v) + a(w) \geq S$. It is well known that G can be partitioned into a clique K and a stable set I . Threshold graphs constitute a very important and well studied class in graph theory and graph algorithms, since they have applications in several fields, such as psychology, parallel processing, scheduling. For this reason, threshold graphs have been defined many times in the literature, with different names (see, e.g. [4,6,7]).

A similar definition describes the class of difference graphs (also known as chain graphs): a graph $G = (V, E)$ is a *difference graph* if there is mapping $a : V \rightarrow \mathbb{R}$ and a positive real number T such that $|a(v)| < T$ for all $v \in V$ and $(v, w) \in E$ if and only if $|a(v) - a(w)| \geq T$.

Difference and threshold graphs are incomparable; difference graphs have been also independently introduced several times, for example in [5,8,12]. For a comprehensive survey on threshold graphs, difference graphs and related topics, see [9].

Let $G = (V, E)$ be a graph whose distinct node-degrees are $\delta_1 < \dots < \delta_m$, and let $\delta_0 = 0$ (even if no node of degree 0 exists). Let $D_i = \{v \in V \text{ s.t. } \deg(v) = \delta_i\}$

¹ Supported in part by the Italian Ministry of Education, University, and Research (MIUR) under PRIN 2012C4E3KT national research project AMANDA Algorithmics for MAssive and Networked DATA

² Email: calamo@di.uniroma1.it

³ Email: monti@di.uniroma1.it

⁴ Email: petreschi@di.uniroma1.it

for $i = 0, \dots, m$; D_i is called i -th box; the sequence D_0, \dots, D_m is called the *degree partition* of G . The notion of degree partition is crucial for understanding threshold graphs as shown in the following lemma:

Lemma 1.1 [9] *Let $G = (V, E)$ be a threshold graph with degree partition D_0, \dots, D_m and whose node set is partitioned into a clique K and a stable set I ; let $x \in D_i$ and $y \in D_j$ be two distinct nodes. The following claims hold:*

- (i) $D_0 \cup \dots \cup D_{\lfloor m/2 \rfloor} = I$ and $D_{\lfloor m/2 \rfloor + 1} \cup \dots \cup D_m = K$;
- (ii) *If $e = (x, y) \notin E$, the graph $G' = (V, E \cup \{e\})$ is a threshold graph if and only if $i + j = m$;*
- (iii) *If $e = (x, y) \in E$, the graph $G' = (V, E \setminus \{e\})$ is a threshold graph if and only if $i + j = m + 1$;*
- (iv) $e = (x, y) \in E$ if and only if $i + j \geq m + 1$.

Given a threshold graph $G = (V, E)$, the pair (a, S) is called *separator*. Although in the definition of a threshold graph its separator has non-negative *real* values, it is common to equivalently require the separator to have non-negative *integral* values (i.e. an integral separator).

An integral separator (a, S) for G is *minimum* if for any other integral separator (a', S') for G we have $S \leq S'$.

2 Main Results

In the following, the results of our work are briefly listed. We first present a very simple algorithm determining the minimum integral separator of a threshold graph. This algorithm runs in linear time w.r.t. the number of different degrees in the graphs, so improving some previous results [10,11], minimizing other weight functions raised by other (equivalent) definitions and running in linear time w.r.t. the number of nodes.

Theorem 2.1 *Let $G = (V, E)$ be a threshold graph with degree partition D_0, \dots, D_m . The pair (a, S) , where $S = m + 1$ and for each node $v \in V$, $a(v) = i$ if $v \in D_i$, is a minimal integral separator of G .*

Proof. First of all, we prove that (a, S) is a separator, i.e. that satisfies the two inequalities of the definition of threshold graphs. Note that for each $v \in V$ it holds $0 \leq a(v) \leq m < S$ thus the pair (a, S) satisfies the first inequality. The second inequality follows from Item 4 in Lemma 1.1. Trivially, (a, S) is integral.

Let us now prove that (a, S) is minimal. By contradiction, let (a, S) be not minimal, and let (a', S') be an integral separator for G such that $S' < S$. Observe that only isolated nodes can have weight equal to zero (indeed, if $(u, v) \in E$ and $a'(u) = 0$ then $a'(u) + a'(v) = a'(v) < S'$ from the first inequality of the definition of threshold graphs, but this contradicts the second inequality of the same definition).

Moreover, notice that two nodes u and v having the same weight necessarily behave in the same way (i.e. for any other node $w \in V$, it holds that $(u, w) \in E$ if and only if $(v, w) \in E$), so nodes having different degrees cannot have the same weight.

All this implies that the function a' on the non isolated nodes assume at least m different strictly positive weights. Thus the first inequality of the definition of threshold graphs implies that $S' \geq m + 1$. The chain $m + 1 = S > S' \geq m + 1$ proves the minimality of (a, S) . \square

The same algorithm can be easily extended to difference graphs.

We then consider the operations of addition/deletion of either an edge or a node with all its incident edges; threshold and difference graphs are not closed w.r.t. these operations but from Lemma 1.1 some characterizations for the added/deleted edge/node can be deduced in order to remain in the same graph class. We show that in these special cases it is possible to modify the node weight function in order to well define the resulting graph. All these operations run in linear time w.r.t. the number of different degrees in the graph. Due to the lack of space we do not describe here these algorithms, that will be detailed in the full version of this paper. We only observe that the node weight function remains minimum even after manipulating the graph through these graph operations.

We study also some binary operations on threshold graphs (disjoint union and join) whose result is in general not in the same graph class anymore, but in a superclass, called threshold signed graphs. A graph $G = (V, E)$ is a *threshold signed graph* if there is a mapping $a : V \rightarrow \mathbb{R}$ and two positive real numbers S and T such that $|a(v)| < \min\{S, T\}$ and $(v, w) \in E$ iff either $|a(v) + a(w)| \geq S$ or $|a(v) - a(w)| \geq T$. Consider $X = \{x \in V \text{ s.t. } a(x) < 0\}$ and $Y = \{x \in V \text{ s.t. } a(x) \geq 0\}$. We can see a threshold signed graph as constituted by two threshold graphs that are connected by a difference graph: the graphs induced by X or by Y are threshold graphs, while the bipartite graph connecting nodes in X to nodes in Y is a difference graph. In order to make easier the exposition, now on we will define a threshold signed graph as $G = (X \cup Y, a, S, T)$.

It is known that the complement graph \bar{G} of a threshold signed graph G is a threshold signed graph. We now describe how to adjust the node weight function and the thresholds when the complement operation is applied to threshold signed graphs.

Theorem 2.2 *Let be given a threshold signed graph $G = (X \cup Y, a, S, T)$. Call $P^* = \max\{S, T\} - \min_{1 \leq i \leq n} \{|a(v_i)|\}$ and $\epsilon = \min_{1 \leq i, j \leq n, (v_i, v_j) \notin E} \{S - |a(v_i) + a(v_j)|, T - |a(v_i) - a(v_j)|\}$.*

Then, its complement is the threshold signed graph $\bar{G} = (X \cup Y, \bar{a}, \bar{S}, \bar{T})$ where:

$$\bar{a}(v) = \begin{cases} -(P^* - a(v)) & \text{if } a(v) \geq 0 \\ (P^* - |a(v)|) & \text{if } a(v) < 0 \end{cases}$$

$$\bar{T} = 2P^* - T + \epsilon \text{ and}$$

$$\bar{S} = 2P^* - S + \epsilon.$$

Proof. We have to prove that the assigned weights and the thresholds are feasible values and that they define exactly the complement of the original graph.

Let us prove first that $|a(\bar{v}_i)| < \min\{\bar{S}, \bar{T}\}$ for each $i = 1, \dots, n$. From the definitions of \bar{S} and \bar{T} we have that both of them are $\geq 2P^* - \max\{S, T\} + \epsilon$ hence,

in particular:

$$(1) \quad \min\{\bar{S}, \bar{T}\} \geq 2P^* - \max\{S, T\} + \epsilon > 2P^* - \max\{S, T\}.$$

On the other hand, $P^* = \max\{S, T\} - \min_{1 \leq i \leq n} \{|a(v_i)|\} \geq \max\{S, T\} - |a(v_i)|$. From this inequality and from (1) we get that $\min\{\bar{S}, \bar{T}\} > P^* - |a(v_i)| = |a(\bar{v}_i)|$.

It remains to prove that $(v_i, v_j) \in \bar{E}$ if and only if either $|a(\bar{v}_i) + a(\bar{v}_j)| \geq S$ or $|a(\bar{v}_i) - a(\bar{v}_j)| \geq T$. We divide the proof into two sub-cases, according to whether (v_i, v_j) belongs to \bar{E} or not.

Assume first that $(v_i, v_j) \in \bar{E}$. This means that $(v_i, v_j) \notin E$, i.e. $|a(v_i) + a(v_j)| < S$ and $|a(v_i) - a(v_j)| < T$. As far as ϵ has been defined, it also holds that:

$$(2) \quad |a(v_i) + a(v_j)| \leq S - \epsilon \text{ and } |a(v_i) - a(v_j)| \leq T - \epsilon.$$

If $a(v_i)$ and $a(v_j)$ have the same sign, then $|a(\bar{v}_i) + a(\bar{v}_j)| = 2P^* - |a(v_i) + a(v_j)| \geq 2P^* - S + \epsilon = \bar{S}$ in view of (2) and of the definition of \bar{S} . In other words, (v_i, v_j) is an \bar{S} -edge.

If $a(v_i)$ and $a(v_j)$ have opposite sign, then $|a(\bar{v}_i) - a(\bar{v}_j)| = 2P^* - |a(v_i) - a(v_j)| \geq 2P^* - T + \epsilon = \bar{T}$ in view of (2) and of the definition of \bar{T} . In other words, (v_i, v_j) is an \bar{T} -edge.

Assume now that $(v_i, v_j) \notin \bar{E}$. This means that $(v_i, v_j) \in E$, i.e. $|a(v_i) + a(v_j)| \geq S$ or $|a(v_i) - a(v_j)| \geq T$.

If $|a(v_i) + a(v_j)| \geq S$ it means that a_i and a_j have the same sign, and hence $a(\bar{v}_i)$ and $a(\bar{v}_j)$ have the same sign, too. So $|a(\bar{v}_i) + a(\bar{v}_j)| = 2P^* - |a(v_i) + a(v_j)| \leq 2P^* - S < 2P^* - S + \epsilon = \bar{S}$, so proving that (v_i, v_j) is not an edge of \bar{G} .

If, finally, $|a(v_i) - a(v_j)| \geq T$ it means that $a(v_i)$ and $a(v_j)$ have different sign, and hence $a(\bar{v}_i)$ and $a(\bar{v}_j)$ have different sign, too. So $|a(\bar{v}_i) - a(\bar{v}_j)| = 2P^* - |a(v_i) - a(v_j)| \leq 2P^* - T < 2P^* - T + \epsilon = \bar{T}$, so proving that also in this case (v_i, v_j) is not an edge of \bar{G} . \square

Observe that, if v is an isolated node of a threshold signed graph it is not restrictive to assume $a(v) = 0$ and, obviously, if $a(v) = 0$ then v is an isolated node. Moreover, it is easy to see that two nodes having the same value of a necessarily behave in the same way. On the other hand, if there are two nodes u and w behaving in the same way and having $a(u) \neq a(w)$ (w.l.o.g. let $a(u) < a(w)$), we can easily modify function a in order to assign them the same value (that is $a(w)$ if u and w are connected and $a(u)$ otherwise). So, the number of different values of function a is a value Δ that is, in general a $O(n)$. It can be proved that Δ is linear w.r.t. the different degrees of the graph, so deducing that the complement of a threshold signed graph can be derived in a time that is proportional to its degree. This is particularly relevant, as recomputing from scratch the weight function would run in quadratic time w.r.t. the number of nodes of the graph [1].

References

- [1] C. Benzaken, P.L. Hammer and D. de Werra. Threshold characterization of graphs with Dilworth number two. *Journal of Graph Theory* 9, 245–267, 1985.
- [2] T. Calamoneri and R. Petreschi. On Pairwise Compatibility Graphs having Dilworth Number k . *Theoretical Computer Science* 547, 82–89, 2014.

- [3] V. Chvátal and P.L. Hammer. Set-packing and threshold graphs, Research Report, Comp. Sci. Dept. University of Waterloo, Canada CORR 73-21, 1973.
- [4] V. Chvátal and P.L. Hammer. Aggregation of inequalities in integer programming, *Annals of Discrete Mathematics* 1, Amsterdam: North-Holland, pp. 145–162, 1977.
- [5] O. Cogis. Ferrers digraphs and threshold graphs. *Discrete Mathematics* 38, 33–46, 1982.
- [6] K. Ecker and S. Zaks. On a graph labelling problem. Bericht 99, Gesellschaft für Mathematik und Datenverarbeitung MBH, Bonn 1977.
- [7] P.B. Henderson and Y. Zalcstein. A graph-theoretic characterization of the PV_{chunk} class of synchronizing primitives. *SIAM Journal of Computing* 6, 88–108, 1977.
- [8] M. Koren. Extreme degree sequences of simple graphs. *Journal of Combinatorial Theory* 2, 253–276, 1972.
- [9] N.V.R. Mahadev and U.N. Peled. Threshold Graphs and Related Topics, *Annals of Discrete Mathematics* 56, North-Holland, Amsterdam, 1995.
- [10] E.T. Ordman. Threshold coverings and resource allocation. *16th Southeastern Conference on Combinatorics, Graph Theory and Computing*, 99–113, 1986.
- [11] J. Orlin. The minimal integral separator of a threshold graph. *Annals of Discrete Mathematics* 1, Amsterdam: North-Holland, pp. 415–419, 1977.
- [12] M. Yannakakis. The complexity of the partial order dimension problem. *SIAM Journal on algebraic and Discrete Methods* 3, 351–358, 1982.