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# Bounds and Fixed-Parameter Algorithms for Weighted Improper Coloring

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## Abstract

We study the weighted improper coloring problem, a generalization of defective coloring. We present some hardness results and in particular we show that weighted improper coloring is not fixed-parameter tractable when parameterized by pathwidth. We generalize bounds for defective coloring to weighted improper coloring and give a bound for weighted improper coloring in terms of the sum of edge weights. Finally we give fixed-parameter algorithms for weighted improper coloring both when parameterized by treewidth and maximum degree and when parameterized by treewidth and precision of edge weights. In particular, we obtain a linear-time algorithm for weighted improper coloring of interval graphs of bounded degree.

*Keywords:* graph coloring, improper coloring, defective coloring, weighted improper coloring, coloring bounds, fixed-parameter algorithms

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# 1 Introduction

Graph coloring is a classic subject of both mathematics and computer science with many practical applications. It was one of Karp’s 21 original  $\mathcal{NP}$ -complete problems. Many generalizations of graph coloring, such as defective coloring, have been studied, but most of those apply to undirected graphs. In this paper we consider weighted improper coloring, a generalization of graph coloring for weighted digraphs. Weighted improper coloring has not received much attention until recently in the context of wireless networks. In some models for wireless networks, such as the SINR model, communication over one wireless link can disturb communication over other wireless links. This disturbance can vary from link to link, and depends on signal strength and other variables of the surrounding environment. Furthermore the disturbance does not need to be symmetric. A scheduling of communications in such a network can be modeled as a weighted improper coloring, and this is where the generalization stems from.

In this paper we provide some new bounds, fixed-parameter algorithms and hardness results for weighted improper coloring, some of which are generalizations of existing results for defective coloring.

## 1.1 Preliminaries

Let  $G = (V, E)$  be an undirected graph. For a vertex  $v \in V$  let  $d(v)$  denote the degree of  $v$  in  $G$ , and  $d_S(v)$  denote the degree of  $v$  in the subgraph induced by a subset  $S \subseteq V$ . A  $k$ -coloring  $c : V \rightarrow \{1, \dots, k\}$  of  $G$  is a partition of the vertex set  $V$  into  $k$  vertex-disjoint subsets, and  $c[v]$  denotes the set of vertices that have color  $c(v)$ . A  $k$ -coloring  $c$  is  $d$ -defective if for each  $v \in V$ ,  $d_{c[v]}(v) \leq d$ . A  $d$ -defective  $k$ -coloring is also called a  $(k, d)$ -defective coloring. Note that ordinary coloring is a special case of defective coloring where  $d = 0$ , so defective coloring is a proper generalization of ordinary coloring.

Let  $G = (V, E, w)$  be a weighted digraph, where  $w : E \rightarrow [0, 1]$  is an associated weight function. For a vertex  $v \in V$  let  $d^-(v) = \sum_{(u,v) \in E} w(u, v)$  denote its weighted indegree, and  $d_S^-(v)$  denote the weighted indegree of  $v$  in the subgraph induced by a subset  $S \subseteq V$ . Let  $\Delta^-$  denote maximum of any weighted indegree.

**Definition 1.1** A  $k$ -coloring  $c$  of a weighted digraph  $G = (V, E, w)$  is a *weighted improper  $k$ -coloring* if for each vertex  $v \in V$ ,  $d_{c[v]}^-(v) < 1$ . The weighted improper chromatic number  $\chi_w(G)$  is the minimum number  $k$  such that  $G$  has a weighted improper  $k$ -coloring with respect to the weight function  $w$ .

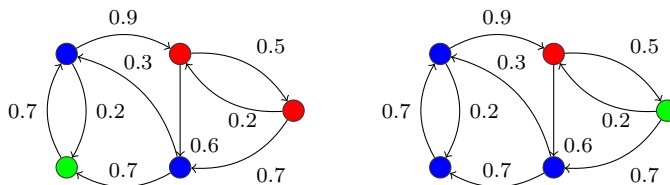


Fig. 1. A valid weighted improper coloring of the graph to the left. The graph on the right has an invalid coloring as the upper left corner vertex has indegree 1 from same-colored neighbors.

Now notice that weighted improper coloring is a generalization of defective coloring, and by extension a generalization of ordinary coloring, as is captured by the following lemma.

**Lemma 1.2** *Defective  $(k, d)$ -coloring can be reduced to weighted improper  $k$ -coloring in polynomial time.*

**Proof.** Let  $G$  be an undirected graph. In a valid defective  $(k, d)$ -coloring of  $G$  any vertex  $v \in V(G)$  can have at most  $d$  adjacent vertices of the same color. Let  $G'$  be the weighted digraph obtained by replacing every edge  $\{u, v\} \in E(G)$  with two directed edges  $(u, v)$  and  $(v, u)$  with weight  $\frac{1}{d+1}$ . Clearly, any vertex  $v' \in V(G')$  cannot have more than  $d$  adjacent vertices of the same color because then we would have  $d_{c[v']}(v') \geq \frac{d+1}{d+1} = 1$ . However,  $v'$  can have up to  $d$  adjacent vertices of the same color because that yields  $d_{c[v']}(v') \leq \frac{d}{d+1} < 1$ . Thus, any valid weighted improper  $k$ -coloring of  $G'$  is also a valid defective  $(k, d)$ -coloring of  $G$  and any invalid weighted improper  $k$ -coloring of  $G'$  is an invalid defective  $(k, d)$ -coloring of  $G$ . The reduction has time complexity  $O(|E(G)|)$  and is thus a polynomial-time reduction.  $\square$

As this reduction does not change the underlying structure of the graph, but only adds weights to it, we get the following corollary.

**Corollary 1.3** *Defective  $(k, d)$ -coloring of any specific class of graphs has a polynomial time reduction to weighted improper  $k$ -coloring of the same class of graphs.*

Finally we define a tree decomposition as follows:

**Definition 1.4** A tree decomposition of a graph  $G = (V, E)$  is a pair  $(X, T)$ , where  $X = \{X_1, \dots, X_n\}$  is a collection of subsets of  $V$ , and  $T$  is a tree whose vertices are the subsets in  $X$ , which we will refer to as super-vertices. Additionally, the following properties must hold:

- (i) if  $v \in V$ , then there exists a subset  $X_i$  such that  $v \in X_i$ ,
- (ii) if  $(u, v) \in E$ , then there exists a subset  $X_i$  such that  $u \in X_i$  and  $v \in X_i$ , and
- (iii) if  $v \in V$  and  $X^v \subseteq X$  is the set of subsets that contain  $v$ , then  $X^v$  forms a connected subtree of  $T$ .

The width of a tree decomposition  $(X, T)$  is  $\max_i |X_i| - 1$ . The treewidth of a graph  $G$  is the minimum width of any tree decomposition of  $G$ . A path decomposition is a tree decomposition  $(X, T)$  where  $T$  is a path graph. The pathwidth of a graph  $G$  is the minimum width of any path decomposition of  $G$ . Finally, path and tree decompositions can be extended to digraphs by using the underlying graph (i.e. by treating directed edges as undirected).

### Previous work

Defective coloring and the related  $t$ -improper coloring were first introduced by Andrews and Jacobsen [1], Harary and Frank [14] and Cowen et al. [10]. Graphs on embeddable surfaces were the main focus of Cowen et al. [10] and Cowen, Goddard and Jeserum [9] and they characterized all  $(k, d)$  such that planar graphs are  $d$ -defective  $k$ -colorable and produced results for surfaces of higher genus. Frick and Henning [12] gave extremal results on the defective chromatic number and Kang and

McDiarmid [17] proved bounds on the growth of the defective chromatic number of random graphs. Other properties than degrees of vertices have been researched regarding defective coloring and Frick [11] gives a good survey on such variations of the problem.

The first to propose this edge weighted variation for undirected graphs were Hofer, Kesselheim and Vöcking [16]. Araujo et al. [3] defined the problem with a variable threshold on the degree of a vertex with respect to its coloring and explored the dual of the problem of finding such a threshold, in addition to producing results on various grid graphs. Similar channel assignment problems have also been considered and modeled as colorings on vertex weighted graphs [6].

Tamura et al. [22] and Archetti et al. [4] present a clear formulation of wireless scheduling of the SINR model systems as directed weighted improper coloring and most of the work has assumed geometric restrictions on the interference, while some general results have been published. Recently Halldórsson and Bang-Jensen [5] gave the essentially tight bound  $\chi_w(G) \leq \lfloor 2\Delta^- + 1 \rfloor$ .

## 2 Hardness

Some hardness results about defective coloring have been established. Cowen and Jeserum [9] proved that  $(2, d)$ -defective coloring is  $\mathcal{NP}$ -complete for  $d \geq 1$ , even for planar graphs, and that  $(3, 1)$ -defective coloring is also  $\mathcal{NP}$ -complete for planar graphs. Furthermore they show that  $(k, d)$ -defective coloring is  $\mathcal{NP}$ -complete for any  $k \geq 3, d \geq 0$ .

These results can be carried over to weighted improper coloring, as we show in the following corollaries.

**Corollary 2.1** *Weighted improper  $k$ -coloring is  $\mathcal{NP}$ -complete for  $k \geq 2$ .*

**Proof.** This follows from Lemma 1.2 and the fact that defective  $(k, d)$ -coloring is  $\mathcal{NP}$ -complete for  $k = 2$  and  $d \geq 1$ , as well as  $k \geq 3$  and  $d \geq 0$ .  $\square$

Note that, as there is a simple polynomial-time algorithm for 2-coloring, it is a bit surprising that defective 2-coloring and weighted improper 2-coloring are  $\mathcal{NP}$ -complete.

**Corollary 2.2** *Weighted improper  $k$ -coloring is  $\mathcal{NP}$ -complete for planar graphs when  $k \in \{2, 3\}$ .*

**Proof.** This follows from Corollary 1.3 and the fact that  $(2, d)$ -defective coloring is  $\mathcal{NP}$ -complete for planar graphs when  $d \geq 1$ , as well as the fact that  $(3, 1)$ -defective coloring is  $\mathcal{NP}$ -complete for planar graphs.  $\square$

Since any planar graph is 4-colorable [2], which is a stricter requirement than being weighted improper 4-colorable, and as 1-colorability is trivial, this, along with Corollary 2.2, gives complete hardness results for weighted improper coloring of planar graphs.

Now consider a graph  $G$ , and say we add any amount of 0-weight edges to this graph to make a new graph  $G'$ . As these 0-weight edges impose no new restrictions,

it is clear that  $G$  is  $k$ -colorable if and only if  $G'$  is  $k$ -colorable. In fact, we can add 0-weight edges until the graph is complete without adding any restrictions, and hence, in a sense, we can always assume that the graph is complete. This observation gives rise to the following lemma.

**Lemma 2.3** *Let  $\mathcal{G}$  be a family of graphs such that for each  $n$  the complete graph  $K_n \in \mathcal{G}$ . Then weighted improper  $k$ -coloring for  $\mathcal{G}$  is  $\mathcal{NP}$ -complete.*

**Proof.** We show this by reducing general weighted improper  $k$ -coloring, which is  $\mathcal{NP}$ -complete by Corollary 2.1, to weighted improper  $k$ -coloring for  $\mathcal{G}$ . Take any weighted digraph  $G = (V, E, w)$ . Create a complete weighted digraph  $G'$  from  $G$  by adding 0-weight edges between pairs of vertices that are not connected by an edge in  $G$ . Clearly  $G$  is  $k$ -colorable if and only if  $G'$  is  $k$ -colorable. Since  $G'$  is a complete graph, we see that  $G' \in \mathcal{G}$ , which concludes the reduction.  $\square$

In particular, this lemma implies the  $\mathcal{NP}$ -completeness of weighted improper coloring for interval graphs,  $C_n \times K_t$ , and  $k$ th powers of paths and cycles, where  $t$  and  $k$  are unbounded. Later we will give fixed-parameter algorithms for these and other graph classes.

This lemma can be further generalized by noticing that vertices with 0-weight edges can also be added to a graph without adding any restrictions, and hence graphs with large enough complete subgraphs can be used instead of complete graphs in the reduction.

**Theorem 2.4** *Weighted improper coloring is  $\mathcal{NP}$ -complete for graphs of bounded pathwidth.*

**Proof.** Given a multiset  $S = \{x_1, \dots, x_n\}$  of positive integers, the partition problem is the task of deciding whether  $S$  can be partitioned into two sets  $S_1, S_2$  such that  $\sum_{x \in S_1} x = \sum_{x \in S_2} x$ . We show that weighted improper coloring is  $\mathcal{NP}$ -complete for graphs of bounded pathwidth by reducing the partition problem to weighted improper coloring of a graph with bounded pathwidth.

We construct an undirected graph  $G = (V, E)$  such that for every integer  $x_i \in S$  we add a corresponding vertex  $v_i$  to  $V$ , along with two additional vertices  $A$  and  $B$ . We add an undirected edge  $\{A, B\}$  with weight 1 and for each vertex  $v_i$  we add two undirected edges  $\{A, v_i\}$  and  $\{v_i, B\}$  to the graph with weights  $w_i = 2x_i/X - \epsilon/|S|$  where  $X = \sum_{x \in S} x$ . The graph is depicted in Figure 2.

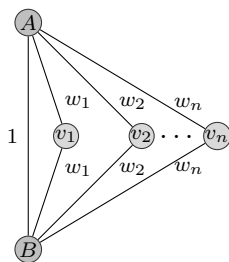


Fig. 2. Subset sum modeled as a weighted improper coloring.

We claim that  $S$  can be partitioned into  $S_1$  and  $S_2$  if and only if there exists a valid weighted improper 2-coloring of  $G$ .

Given a partition of  $S$  into  $S_1, S_2$  we assign to every vertex  $v_i$  corresponding to an integer  $x_i \in S_1$  the color 1 and conversely we assign the color 2 to every vertex corresponding to an integer  $x_i \in S_2$ . In addition we assign 1 to  $A$  and 2 to  $B$ . For every  $v_i$  we have  $d_{c[v_i]}(v_i) = w_i = 2x_i/X - \epsilon/|S| < 1$  as a single integer cannot exceed half of the total sum of the integers, otherwise the partition would not be valid. For  $A$  we get

$$d_{c[A]}(A) = \sum_{x_i \in S_1} w_i = \frac{2}{X} \sum_{x_i \in S_1} x_i - \frac{|S_1|}{|S|} \epsilon = \frac{2}{X} \cdot \frac{X}{2} - \frac{|S_1|}{|S|} \epsilon = 1 - \frac{|S_1|}{|S|} \epsilon < 1.$$

The same argument holds for  $B$ . Hence the partition yields a valid weighted improper 2-coloring of  $G$ .

Given a valid weighted improper coloring of  $G$ , we let  $S_1$  be all those vertices in the same color class as  $A$  and  $S_2$  be all those in the same color class as  $B$ . Then

$$\sum_{x_i \in S_1} x_i = \sum_{v_i \in c[A]} \frac{X}{2} \left( w_i + \frac{\epsilon}{|S|} \right) = \frac{X}{2} \left( \sum_{v_i \in c[A]} w_i + \frac{|S_1|}{|S|} \epsilon \right) \leq \frac{X}{2} \left( 1 - \epsilon + \frac{|S_1|}{|S|} \epsilon \right) \leq \frac{X}{2},$$

and the same holds for  $S_2$ . For both  $S_1, S_2$  we get  $\sum_{x \in S_i} x \leq \frac{X}{2}$  and by definition  $\sum_{x \in S_1} x + \sum_{x \in S_2} x = X$ , hence equality holds.



Fig. 3. Path decomposition of the graph constructed from an instance of the partition problem.

The graph  $G$  has pathwidth 2 as can be seen by an optimal path decomposition of  $G$  depicted in Figure 3. Using the fact that the partition problem is  $\mathcal{NP}$ -complete [15], we conclude the proof.  $\square$

### 3 Bounds

Prohibiting edges of weight 1 can make a difference when coloring weighted graphs. The 3-regular graph in Figure 4 has weighted improper chromatic number 3, even though every vertex has an incident edge with weight less than 1. However, 3-regular graphs with no edges of weight 1 have weighted improper chromatic number at most 2, as is shown by the following lemma.

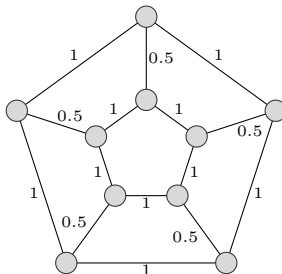


Fig. 4. A 3-regular graph where each vertex has an incident edge with weight less than 1.

**Lemma 3.1** *If  $G = (V, E, w)$  is a weighted graph such that  $w(e) < 1$  for every  $e \in E$  and every vertex has at most 3 neighbors, then  $\chi_w(G) \leq 2$ .*

**Proof.** Start with any 2-coloring  $c : V \rightarrow \{1, 2\}$  of  $G$ . While we have some vertex with at least two same-colored neighbors, pick such a vertex  $v$  and flip the color of  $v$  to the opposite color. This step reduces the number of monochromatic edges by at least 1 and as there are at most  $|E|$  monochromatic edges to begin with, the procedure halts after a finite number of steps. When the procedure halts, every vertex has at most one incident monochromatic edge and as the weights are less than 1, the degree of the vertex with respect to its color is less than 1.  $\square$

We can extend the results of Lovász [20] to our version of the problem. It is clear that  $\frac{\chi(G)}{(d+1)} \leq \chi_d(G)$  [18], where  $\chi_d(G)$  is the  $d$ -defective chromatic number. Similar results hold for undirected weighted improper coloring.

**Lemma 3.2** *If  $G = (V, E, w)$  is an undirected weighted graph with the minimum positive edge weight  $w_{\min}$ , then*

$$\left\lceil \frac{\chi(G)}{\left\lfloor \frac{1-\epsilon}{w_{\min}} \right\rfloor + 1} \right\rceil \leq \chi_w(G).$$

**Proof.** We start by removing all edges  $e \in E$  with  $w(e) = 0$ . Given a weighted improper coloring of  $G$  with  $\chi_w(G)$  colors, each vertex will have at most  $\left\lfloor \frac{1-\epsilon}{w_{\min}} \right\rfloor$  neighbors with the same color. Hence, by Brooks' theorem, each of the subgraphs induced by the coloring can be properly colored with at most  $\left\lfloor \frac{1-\epsilon}{w_{\min}} \right\rfloor + 1$  colors. Therefore  $\chi(G) \leq \left( \left\lfloor \frac{1-\epsilon}{w_{\min}} \right\rfloor + 1 \right) \chi_w(G)$ , which yields the lemma.  $\square$

**Theorem 3.3** *If  $G = (V, E, w)$  is a weighted digraph, where the underlying graph of  $G$  has maximum degree  $\hat{\Delta}$  and maximum edge weight  $w_{\max}$ , then*

$$\chi_w(G) \leq \left\lceil \frac{\hat{\Delta}}{\left\lfloor \frac{1-\epsilon}{w_{\max}} \right\rfloor + 1} \right\rceil + 1.$$

**Proof.** We construct a weighted undirected graph  $G' = (V', E' = \{\{u, v\} : (u, v) \in E \text{ or } (v, u) \in E\}, w')$  where  $w'(e) = w_{\max}$  for every  $e \in E'$ . Clearly  $\chi_w(G) \leq \chi_{w'}(G')$  as we are adding edges and increasing the weights. Notice that each  $v \in V'$  can have at most  $\left\lfloor \frac{1-\epsilon}{w_{\max}} \right\rfloor$  same-colored neighbors in a valid weighted improper coloring. Let  $k = \left\lceil \hat{\Delta} / \left\lfloor \frac{1-\epsilon}{w_{\max}} \right\rfloor + 1 \right\rceil + 1$  and start with any  $k$ -coloring of  $G'$ .

Say there exists a vertex  $v$  with more than  $\left\lfloor \frac{1-\epsilon}{w_{\max}} \right\rfloor$  neighbors of the same color. First assume that  $v$  has more than  $\left\lfloor \frac{1-\epsilon}{w_{\max}} \right\rfloor$  neighbors of every color class. But then

$$d_{G'}(v) \geq \left( \left\lfloor \frac{1-\epsilon}{w_{\max}} \right\rfloor + 1 \right) \left\lceil \frac{\hat{\Delta}}{\left\lfloor \frac{1-\epsilon}{w_{\max}} \right\rfloor + 1} \right\rceil \geq \hat{\Delta} + \frac{\hat{\Delta}}{\left\lfloor \frac{1-\epsilon}{w_{\max}} \right\rfloor + 1},$$

which contradicts the fact that  $\hat{\Delta}$  is the maximum degree in the underlying graph of  $G$ . Hence there exists a color class where  $v$  has at most  $\lfloor \frac{1-\epsilon}{w_{\max}} \rfloor$  neighbors, and we will change the color of  $v$  to the color of any such color class.

We can repeat the previous procedure while there exists a vertex with more than  $\lfloor \frac{1-\epsilon}{w_{\max}} \rfloor$  neighbors of the same color. Now consider the monochromatic edges in the graph, which are at most  $|E|$  to begin with. Every time we change the color of a vertex during this process, the number of monochromatic edges decreases by at least one. Hence this process must terminate, and at that time each vertex has at most  $\lfloor \frac{1-\epsilon}{w_{\max}} \rfloor$  same-colored neighbors, so we have a valid  $k$ -coloring.  $\square$

As the previous proof is constructive, we get the following corollary:

**Corollary 3.4** *For a weighted digraph  $G = (V, E, w)$  with maximum weight  $w_{\max}$  and maximum degree  $\hat{\Delta}$  in the underlying graph, a weighted improper  $\left(\left\lceil \frac{\hat{\Delta}}{\lfloor \frac{1-\epsilon}{w_{\max}} \rfloor + 1} \right\rceil + 1\right)$ -coloring can be found in  $O(\hat{\Delta}|E|)$  time.*

Halldórsson and Bang-Jensen [5] proved the essentially tight bound of  $\chi_w(G) \leq \lfloor 2\hat{\Delta}^{-1} + 1 \rfloor$  in terms of maximum weighted indegree. This bound is used in the following theorem to give a bound in terms of sum of edge weights.

**Theorem 3.5** *If  $G = (V, E, w)$  is a weighted digraph and  $W = \sum_{e \in E} w(e)$ , then  $\chi_w(G) \leq 2 \lfloor \sqrt{2W} \rfloor + 1$ .*

**Proof.** Let  $t$  denote the number of vertices  $v$  of  $G$  with  $d^-(v) \geq \sqrt{W/2}$ . Since the sum of indegrees of those vertices is at least  $t \cdot \sqrt{W/2} \leq W$ , we get that  $t \leq \sqrt{2W}$ . Let  $G'$  be the induced subgraph of the vertices with indegree at least  $\sqrt{W/2}$ . By [5] the rest of the graph  $G \setminus G'$  has chromatic number  $\chi_w(G \setminus G') \leq \lfloor 2\sqrt{W/2} + 1 \rfloor = \lfloor \sqrt{2W} + 1 \rfloor$ . By coloring  $G'$  with  $\lfloor \sqrt{2W} \rfloor$  colors, we get a coloring using at most  $\lfloor \sqrt{2W} \rfloor + \lfloor \sqrt{2W} + 1 \rfloor$  colors, and the theorem follows.  $\square$

## 4 Fixed-Parameter Algorithms

Both ordinary graph coloring and defective coloring are fixed-parameter tractable when parameterized by treewidth [19,21]. Thus it is natural to ask whether the same holds true for weighted improper coloring. Theorem 2.4 shows that this is not the case.

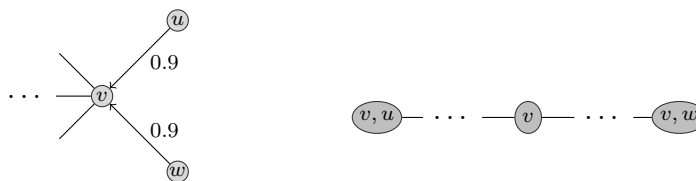


Fig. 5. A weighted digraph and its tree decomposition.

Figure 5 shows a small part of a graph (on the left) and what the relation between these vertices might look like in a tree decomposition of the same graph (on the right). The classic fixed-parameter algorithm for ordinary coloring would



at some point visit the vertex corresponding to the subset  $\{v, u\}$  and choose colors for  $v$  and  $u$ . Then it would proceed down the tree decomposition, possibly visiting multiple vertices, and then finally arrive at the vertex corresponding to the subset  $\{v, w\}$ . At that point  $w$  needs to be assigned some color, and it is immediately clear that it cannot be the same color as is assigned to  $v$ . On the other hand, if this was weighted improper coloring, then it may or may not be valid to assign  $v$ 's assigned color to  $w$ . This depends on which neighbors of  $v$  have the same color as  $v$  does. In this example it depends on the color of  $u$ , and possibly other vertices. This shows that valid colors for a vertex may depend on multiple decisions that are not local to its closest ancestors in the tree decomposition, possibly giving some intuition into why weighted improper coloring is not fixed-parameter tractable when parameterized only by treewidth.

However, we now show that weighted improper coloring is fixed-parameter tractable when parameterized by treewidth and either the maximum indegree in a graph or the precision of edge weights. As a first step towards this, we present the following fact.

**Fact 4.1** [8, Theorem 6] *The chromatic number of a graph  $G$  of treewidth  $k$  is at most  $k + 1$ .*

As ordinary coloring is a stricter requirement than weighted improper coloring, it follows that the weighted improper chromatic number of a graph  $G$  of treewidth  $k$  is also at most  $k + 1$ .

The following algorithms operate on an optimal tree decomposition, which can be constructed in linear time (when the treewidth is bounded) by Bodlaender [7].

#### 4.1 Bounded Treewidth and Bounded Maximum Indegree

If we additionally restrict the maximum unweighted indegree of the graph, then it is possible to keep track of the colors of all vertices and all their in-neighbors within a given super-vertex in the tree decomposition, which is exactly the information needed to check if the coloring restrictions for a given vertex within that super-vertex are fulfilled.

Now consider any weighted digraph  $G = (V, E, w)$  with bounded treewidth  $k$  and bounded maximum indegree. Algorithm 1 is an adaptation of the classic fixed-parameter algorithm for ordinary graph coloring. It additionally keeps track of colors of the in-neighbors of the vertices being colored to be able to check weight restrictions.

A call to  $\text{COLOR}(X_i, \bar{c})$  returns the minimum  $l$  such that the subgraph induced by the vertices contained in the super-vertices in the subtree rooted at  $X_i$  is  $l$ -colorable with respect to the partial coloring  $\bar{c}$ . To answer the query  $\text{COLOR}$  tries all possible colorings  $c$  of the vertices and their neighbors using the same colors for those vertices that have already been colored with the coloring  $\bar{c}$ . Then it recursively calls itself to answer further subproblems.

In order to argue the correctness of this algorithm, there are essentially three properties that need to be shown:

- the assignment of colors to vertices is consistent, that is, exactly one color is

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**Algorithm 1** A fixed-parameter algorithm for graphs of bounded treewidth and bounded maximum indegree.

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1: function COLOR( $X_i, \bar{c}$ )
2: begin
3:   Let  $X_p$  be the parent of  $X_i$  in  $T$ , or the empty set if  $X_i$  is the root
4:   Let  $V_i$  be the set of vertices in  $X_i$  and all their in-neighbors, and let  $V_p$ 
   be defined similarly for  $X_p$ 
5:    $\chi \leftarrow +\infty$ 
6:   for each coloring  $c : V_i \rightarrow \{1, \dots, k+1\}$  with  $c(v) = \bar{c}(v)$  for  $v \in V_i \cap V_p$  do
7:     if  $d_{c[v]}^-(v) < 1$  for each  $v \in X_i$  then
8:        $\chi' \leftarrow$  highest index of a color used by  $c$ 
9:       for each child  $X_j$  of  $X_i$  do
10:         $\chi' \leftarrow \max(\chi', \text{COLOR}(X_j, c))$ 
11:       end for
12:        $\chi \leftarrow \min(\chi, \chi')$ 
13:     end if
14:   end for
15:   return  $\chi$ 
16: end

17: Construct a tree decomposition  $(X, T)$  of  $G$  of width at most  $k$ 
18: Root the tree  $T$  at super-vertex  $X_1$ 
19: return COLOR( $X_1, \emptyset$ )

```

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assigned to a given vertex, and this color is used throughout the duration of the algorithm,

- any coloring that the algorithm discovers is valid, and
- any valid coloring can be discovered by the algorithm.

We show that these properties hold in Section A.1 of Appendix A in the extended version of this paper [13]. This gives us the following theorem.

**Theorem 4.2** *Weighted improper coloring is fixed-parameter tractable when parameterized by treewidth and maximum indegree.*

**Proof.** Lemmas A.4 and A.5 show that the colorings discovered by Algorithm 1 are exactly the valid colorings using at most  $k+1$  colors. In particular, as the weighted improper chromatic number is at most  $k+1$  by Lemma 4.1, it will discover all colorings that have the minimum number of colors. And as the algorithm returns the minimum number of colors in any coloring found, it will return the weighted improper chromatic number of the input graph  $G$ .

Algorithm 1 has exponential time complexity, as it is essentially a backtracking algorithm that explores all valid colorings of at most  $k+1$  colors. Now notice that the results of the COLOR function only depend on the values of its input parameters. Also notice that the call hierarchy of the recursive calls is acyclic. Therefore the results of these function calls can be computed efficiently using dynamic programming. That is, by computing the results of the function calls for all possible input parameters, storing all results, and then re-using the stored results

when needed. Let  $n = |V(G)|$ ,  $k$  be the treewidth and  $\hat{\Delta}^-$  denote the maximum unweighted indegree of  $G$ . As there are  $(k+1)^{(k+1)\hat{\Delta}^-}$  possible maps from a set of  $(k+1)\hat{\Delta}^-$  elements to a set of  $k+1$  elements, there are  $O\left(n(k+1)^{(k+1)\hat{\Delta}^-}\right)$  possible input parameters to the COLOR function. As the result for each of them can be computed in  $O\left(n(k+1)^{(k+1)\hat{\Delta}^-}\right)$  time, this gives us total time complexity  $O\left(n^2(k+1)^{2(k+1)\hat{\Delta}^-}\right)$ . Although unnecessary for this result, noticing that each super-vertex is actually only referenced twice reduces the time complexity down to  $O\left(n(k+1)^{2(k+1)\hat{\Delta}^-}\right)$ . This time complexity along with the correctness of the algorithm proves that weighted improper coloring is fixed-parameter tractable when parameterized by treewidth and maximum indegree.  $\square$

The algorithm has memory complexity  $O\left(n(k+1)^{(k+1)\hat{\Delta}^-}\right)$ . It may also be of interest that, using common dynamic programming techniques, it is possible to construct an optimal coloring, or count the number of optimal colorings.

There are many graphs of bounded treewidth and bounded maximum degree, and this algorithm is linear when applied to these graphs. This includes bounded degree interval graphs,  $C_n \times K_t$  where  $t$  is bounded, and  $k$ th powers of paths and cycles where  $k$  is bounded. Remember that it is  $\mathcal{NP}$ -complete to weighted improper color the unbounded versions of these graph classes by Lemma 2.3.

#### 4.2 Bounded Treewidth and Bounded Precision Weights

Another possibility is to additionally restrict the precision of weights in the graph. Consider the process of assigning colors to vertices one at a time. Initially, when no vertex has been colored, all vertices have weighted indegree zero from same-colored vertices, and are allowed to have an additional weighted indegree of  $1 - \epsilon$  of same-colored vertices. We will call this the budget of the vertex. As more vertices are assigned colors, the budget of a given vertex gradually decreases. If the weights have bounded precision, then it is possible to keep track of the budgets in such a process.

Now consider any weighted digraph  $G = (V, E, w)$  with bounded treewidth  $k$  and edge weights with bounded precision of  $b$  bits. Let  $R$  denote the set of  $b$ -bit fixed precision real numbers. Algorithm 2, similar to Algorithm 1, is an adaptation of the classic fixed-parameter algorithm for ordinary graph coloring, but instead of the coloring of the neighbors, it keeps track of vertex budgets to check weight restrictions.

A call to  $\text{COLOR}(X_i, \bar{c}, \bar{r})$  returns the minimum  $l$  such that the subgraph induced by the vertices contained in the super-vertices in the subtree rooted at  $X_i$  is  $l$ -colorable, provided that

- the partial coloring given by the parameter  $\bar{c}$  is the coloring done at the parent of  $X_i$  (which could share some vertices with  $X_i$ ), and
- the second parameter  $\bar{r}$  gives the remaining budgets of the vertices in  $X_i$ 's parent.

To answer this query COLOR tries all possible colorings  $c$  of the vertices in  $X_i$  while maintaining the colors given by  $\bar{c}$ . However, it cannot directly call itself recursively to solve further subproblems. The reason is that the budget of a given

vertex is not independent amongst the children super-vertices. In order to break up this dependency, the DISTRIBUTE function distributes the budget of a given vertex amongst the children super-vertices. This is done by recursively going over all child super-vertices of super-vertex  $X_p$  and deciding how much of the budget may be spent in the subtree rooted at the  $i$ -th child. Then COLOR can be called with this budget,  $r$ , on the  $i$ -th child to solve further subproblems.

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**Algorithm 2** A fixed-parameter algorithm for graphs of bounded treewidth and bounded precision weights.

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1: function DISTRIBUTE( $X_p, \bar{c}, \bar{r}, i$ )
2: begin
3:   if  $X_p$  has less than  $i$  children then
4:     return 0
5:   else
6:     Let  $X_i$  be the  $i$ -th child of  $X_p$ 
7:      $\chi \leftarrow +\infty$ 
8:     for each possible  $r : X_p \rightarrow R$  where  $0 \leq r(v) \leq \bar{r}(v)$  do
9:        $\chi' \leftarrow \max(\text{DISTRIBUTE}(X_p, \bar{c}, (\bar{r} - r), i + 1), \text{COLOR}(X_i, \bar{c}, r))$ 
10:       $\chi \leftarrow \min(\chi, \chi')$ 
11:    end for
12:    return  $\chi$ 
13:  end if
14: end

15: function COLOR( $X_i, \bar{c}, \bar{r}$ )
16: begin
17:   Let  $X_p$  be the parent of  $X_i$  in  $T$ , or the empty set if  $X_i$  is the root
18:    $\chi \leftarrow +\infty$ 
19:   for each coloring  $c : V_i \rightarrow \{1, \dots, k + 1\}$  with  $c(v) = \bar{c}(v)$  for  $v \in V_i \cap V_p$  do
20:     Let  $r : V_i \rightarrow R$  such that  $r(v) = \bar{r}(v)$  if  $v \in V_p$  and  $r(v) = 1 - \epsilon$  otherwise
21:     for each  $(u, v) \in E(G)$  do
22:       if  $\{u, v\} \subseteq X_i$  and  $(u \notin X_p \text{ or } v \notin X_p)$  then
23:          $r(v) \leftarrow r(v) - w(u, v)$ 
24:       end if
25:     end for
26:     if  $r(v) \geq 0$  for each  $v \in X_i$  then
27:        $\chi \leftarrow \min(\chi, \text{DISTRIBUTE}(X_i, c, r, 1))$ 
28:     end if
29:   end for
30:   return  $\chi$ 
31: end

32: Construct a tree decomposition  $(X, T)$  of  $G$  of width at most  $k$ 
33: Root the tree  $T$  at super-vertex  $X_1$ 
34: return COLOR( $X_1, \emptyset, \emptyset$ )

```

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In order to argue the correctness of this algorithm the same three properties as

with Algorithm 1 need to be shown. We show that these properties hold in Section A.2 of Appendix A in the extended version of this paper [13]. This gives us the following theorem.

**Theorem 4.3** *Weighted improper coloring is fixed-parameter tractable when parameterized by treewidth and precision of weights.*

**Proof.** Lemmas A.8 and A.9 show that the colorings discovered by Algorithm 2 are exactly the valid colorings using at most  $k + 1$  colors. In particular, as the weighted improper chromatic number is at most  $k + 1$  by Lemma 4.1, it will discover all colorings that have the minimum number of colors. And as the algorithm returns the minimum number of colors in any coloring found, it will return the weighted improper chromatic number of the input graph  $G$ .

Algorithm 2 has exponential time complexity, as it is essentially a backtracking algorithm that explores all valid colorings of at most  $k + 1$  colors. Now notice that the results of the DISTRIBUTE and COLOR functions only depend on the values of their input parameters. Also notice that the call hierarchy of the recursive calls is acyclic. Therefore the results of these function calls can be computed efficiently using dynamic programming. That is, by computing the results of the function calls for all possible input parameters, storing all results, and then reusing the stored results when needed. Let  $n = |V(G)|$ ,  $k$  be the treewidth, and  $b$  be the precision of weights (i.e. the number of bits used to represent them). As there are  $(k + 1)^{k+1}$  possible maps from a set of  $k + 1$  elements to a set of  $k + 1$  elements, there are  $O(n^2(k + 1)^{k+1}2^{b(k+1)})$  possible input parameters to the DISTRIBUTE function. The result for each of them can be computed in  $O(2^{b(k+1)})$  time. Similarly we can see that there are  $O(n(k + 1)^{k+1}2^{b(k+1)})$  possible input parameters to the COLOR function and the result for each of them can be computed in  $O(n^2(k + 1)^{k+1})$  time. This gives us total time complexity  $O(n^2(k + 1)^{k+1}2^{2b(k+1)} + n^3(k + 1)^{2(k+1)}2^{b(k+1)})$ . This time complexity along with the correctness of the algorithm proves that weighted improper coloring is fixed-parameter tractable when parameterized by treewidth and precision of weights.  $\square$

## 5 Conclusion

In this paper we defined the weighted improper coloring problem and presented some hardness results for it. In particular we showed that weighted improper coloring is not fixed-parameter tractable when pathwidth is fixed. We generalized some bounds for defective coloring to weighted improper coloring, and used a bound by Halldórsson and Bang-Jensen [5] to derive a bound for weighted improper coloring in terms of the sum of edge weights. As this bound is not tight, it would be interesting to find a tight bound in terms of the sum of edge weights.

We also showed that 3-regular graphs that have edge weights strictly less than 1 are 2-colorable, and that two colors may not be sufficient when each vertex has at most two incident weight-1 edges. It would be interesting to find necessary and sufficient conditions on edge weights for a 3-regular graph to be 2-colorable. In particular we conjecture that sub-cubic graphs with at most one incident weight-1 edge are 2-colorable.

We gave fixed-parameter algorithms for weighted improper coloring when either treewidth and maximum degree are fixed or when treewidth and precision of edge weights are fixed. These algorithms also imply a linear-time algorithm for certain graph classes such as interval graphs with bounded degree. A combination of rounding edge weights with the fixed-parameter algorithm for bounded treewidth and bounded precision weights might imply an approximation algorithm for weighted improper coloring of graphs of bounded treewidth.

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